Completing the Square

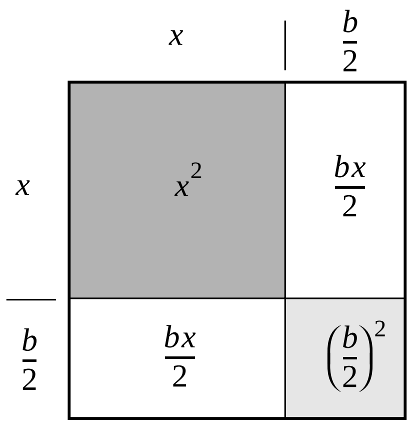
Here's a beautiful little piece of high school algebra, which, as it turns out, can only be truly appreciated long after high school, when your hormones have reached a low enough level that anything not directly related to the opposite sex can engage your interest at all. Chances are that, by then, you've forgotten most of what you learned. If you're lucky, someday when you have a child who is a high school algebra student, you get another opportunity to see stuff like this. Consider the following second order polynomial.

$$x^2 + bx + c$$

What does c have to be in order for the expression to be a square that can be rewritten as

$$(x + d)^2 ?$$

In other words, given $x^2 + bx$, how do we "complete the square"? Here comes the fun part, for those of us who find geometry more intuitive than algebra. The figure is of a square that has sides that are $x + \frac{b}{2}$ long.



The area of each rectangle in the square is shown inside the rectangle. Obviously, the area of the large square is

$$\left(x + \frac{b}{2}\right)^2,$$

the form we want, and equal to the sum of the four rectangular parts,

$$x^2 + \frac{bx}{2} + \frac{bx}{2} + \left(\frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2,$$

which is the "completed square", where

$$c = \left(\frac{b}{2}\right)^2.$$

Cool, huh?

Solving a Quadratic Equation

Now let's suppose we wanted to solve the following equation.

$$ax^2 + bx + c = 0$$

(Believe it or not, sometimes people who are no longer algebra students actually want to do this. Well, of course, most high school algebra students don't really care what the solution is, but might have some ulterior motive, e.g., passing a test to pass the course to graduate from high school to go to college where there should be plenty of freedom to pursue interests in the opposite sex...)

First, divide by a and isolate the incomplete square on the left side of the equation.

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

Then, add a term that completes the square on the left side.

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Now, rewrite the left side as the square of a binomial, which was the point of completing the square.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Finally, take the square root of both sides and grind through solving for x,

$$x + \frac{b}{2a} = \pm\sqrt{-\frac{c}{a} + \left(\frac{b}{2a}\right)^2}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{(2a)^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ending with the *quadratic formula*.

Never again will you have to look up the quadratic formula.